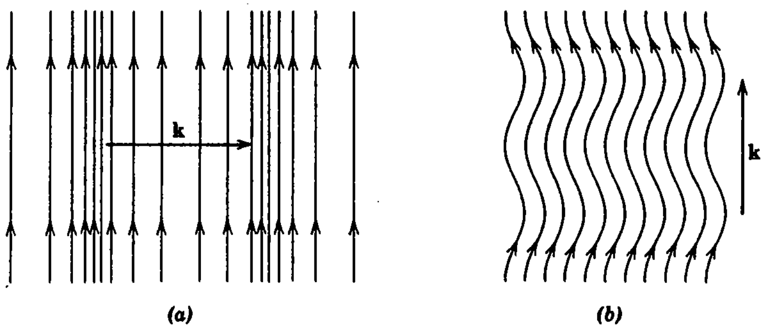

Figure 7.12 Magnetohydrodynamic waves.

compared with the sound speed of 1.45×10^3 m/s. At all laboratory field strengths the Alfvén velocity is much less than the speed of sound. In astrophysical problems, on the other hand, the Alfvén velocity can become very large because of the much smaller densities. In the sun's photosphere, for example, the density is of the order of 10^{-4} kg/m^3 ($\sim 6 \times 10^{22}$ hydrogen atoms/m^3) so that $v_A \simeq 10^5\, B(\text{T})$ m/s. Solar magnetic fields appear to be of the order of 1 or 2×10^{-4} T at the surface, with much larger values around sunspots. For comparison, the velocity of sound is of the order of 10^4 m/s in both the photosphere and the chromosphere.

The magnetic fields of these different waves can be found from the third equation in (7.71):

$$
\mathbf{B}_1 = \begin{cases} \dfrac{k}{\omega} v_1 \mathbf{B}_0 & \text{for } \mathbf{k} \perp \mathbf{B}_0 \\ 0 & \text{for the longitudinal } \mathbf{k} \parallel \mathbf{B}_0 \\ -\dfrac{k}{\omega} B_0 \mathbf{v}_1 & \text{for the transverse } \mathbf{k} \parallel \mathbf{B}_0 \end{cases} \tag{7.78}
$$

The magnetosonic wave moving perpendicular to $\mathbf{B}_0$ causes compressions and rarefactions in the lines of force without changing their direction, as indicated in Fig. 7.12*a*. The Alfvén wave parallel to $\mathbf{B}_0$ causes the lines of force to oscillate back and forth laterally (Fig. 7.12*b*). In either case the lines of force are "frozen in" and move with the fluid.

Inclusion of the effects of fluid viscosity, finite, not infinite, conductivity, and the displacement current add complexity to the analysis. Some of these elaborations are treated in the problems.

7.8 *Superposition of Waves in One Dimension; Group Velocity*

In the preceding sections plane wave solutions to the Maxwell equations were found and their properties discussed. Only monochromatic waves, those with a definite frequency and wave number, were treated. In actual circumstances such idealized solutions do not arise. Even in the most monochromatic light source or the most sharply tuned radio transmitter or receiver, one deals with a finite (although perhaps small) spread of frequencies or wavelengths. This spread may originate in the finite duration of a pulse, in inherent broadening in the source, or in many other ways. Since the basic equations are linear, it is in principle an elementary matter to make the appropriate linear superposition of solutions with different frequencies. In general, however, several new features arise.

1. If the medium is dispersive (i.e., the dielectric constant is a function of the frequency of the fields), the phase velocity is not the same for each frequency component of the wave. Consequently different components of the wave travel with different speeds and tend to change phase with respect to one another.
2. In a dispersive medium the velocity of energy flow may differ greatly from the phase velocity, or may even lack precise meaning.
3. In a dissipative medium, a pulse of radiation will be attenuated as it travels with or without distortion, depending on whether the dissipative effects are or are not sensitive functions of frequency.

The essentials of these dispersive and dissipative effects are implicit in the ideas of Fourier series and integrals (Section 2.8). For simplicity, we consider scalar waves in only one dimension. The scalar amplitude $u(x, t)$ can be thought of as one of the components of the electromagnetic field. The basic solution to the wave equation has been exhibited in (7.6). The relationship between frequency ω and wave number k is given by (7.4) for the electromagnetic field. Either ω or k can be viewed as the independent variable when one considers making a linear superposition. Initially we will find it most convenient to use k as an independent variable. To allow for the possibility of dispersion we will consider ω as a general function of k:

$$\omega = \omega(k) \tag{7.79}$$

Since the dispersive properties cannot depend on whether the wave travels to the left or to the right, ω must be an even function of k, $\omega(-k) = \omega(k)$. For most wavelengths ω is a smoothly varying function of k. But, as we have seen in Section 7.5, at certain frequencies there are regions of "anomalous dispersion" where ω varies rapidly over a narrow interval of wavelengths. With the general form (7.79), our subsequent discussion can apply equally well to electromagnetic waves, sound waves, de Broglie matter waves, etc. For the present we assume that k and $\omega(k)$ are real, and so exclude dissipative effects.

From the basic solutions (7.6) we can build up a general solution of the form

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx - i\omega(k)t} \, dk \tag{7.80}$$

The factor $1\sqrt{2\pi}$ has been inserted to conform with the Fourier integral notation of (2.44) and (2.45). The amplitude $A(k)$ describes the properties of the linear superposition of the different waves. It is given by the transform of the spatial amplitude $u(x, t)$, evaluated at $t = 0$*:

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, 0) e^{-ikx} \, dx \tag{7.81}$$

If $u(x, 0)$ represents a harmonic wave e^{ik_0x} for all x, the orthogonality relation (2.46) shows that $A(k) = \sqrt{2\pi}\, \delta(k - k_0)$, corresponding to a monochromatic

*The following discussion slights somewhat the initial-value problem. For a second-order differential equation we must specify not only $u(x, 0)$ but also $\partial u(x, 0)/\partial t$. This omission is of no consequence for the rest of the material in this section. It is remedied in the following section.

traveling wave $u(x, t) = e^{ik_0x - i\omega(k_0)t}$, as required. If, however, at $t = 0$, $u(x, 0)$ represents a finite wave train with a length of order Δx, as shown in Figure 7.13, then the amplitude $A(k)$ is not a delta function. Rather, it is a peaked function with a breadth of the order of Δk, centered around a wave number k_0, which is the dominant wave number in the modulated wave $u(x, 0)$. If Δx and Δk are defined as the rms deviations from the average values of x and k [defined in terms of the *intensities* $|u(x, 0)|^2$ and $|A(k)|^2$], it is possible to draw the general conclusion:

$$\Delta x \, \Delta k \geq \tfrac{1}{2} \tag{7.82}$$

The reader may readily verify that, for most reasonable pulses or wave packets that do not cut off too violently, Δx times Δk lies near the lower limiting value in (7.82). This means that short wave trains with only a few wavelengths present have a very wide distribution of wave numbers of monochromatic waves, and conversely that long sinusoidal wave trains are almost monochromatic. Relation (7.82) applies equally well to distributions in time and in frequency.

The next question is the behavior of a pulse or finite wave train in time. The pulse shown at $t = 0$ in Fig. 7.13 begins to move as time goes on. The different frequency or wave-number components in it move at different phase velocities. Consequently there is a tendency for the original coherence to be lost and for the pulse to become distorted in shape. At the very least, we might expect it to propagate with a rather different velocity from, say, the average phase velocity of its component waves. The general case of a highly dispersive medium or a very sharp pulse with a great spread of wave numbers present is difficult to treat. But the propagation of a pulse which is not too broad in its wave-number spectrum, or a pulse in a medium for which the frequency depends weakly on wave number, can be handled in the following approximate way. The wave at time t is given by (7.80). If the distribution $A(k)$ is fairly sharply peaked around some value k_0, then the frequency $\omega(k)$ can be expanded around that value of k:

$$\omega(k) = \omega_0 + \left.\frac{d\omega}{dk}\right|_0 (k - k_0) + \cdots \tag{7.83}$$

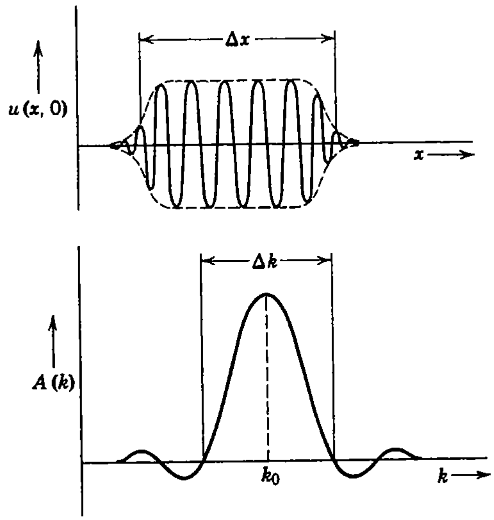

Figure 7.13 A harmonic wave train of finite extent and its Fourier spectrum in wave number.

and the integral performed. Thus

$$u(x, t) \simeq \frac{e^{i[k_0(d\omega/dk)|_0 - \omega_0]t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{i[x-(d\omega/dk)|_0 t]k}\, dk \tag{7.84}$$

From (7.81) and its inverse it is apparent that the integral in (7.84) is just $u(x', 0)$, where $x' = x - (d\omega/dk)|_0\, t$:

$$u(x, t) \simeq u\left(x - t\left.\frac{d\omega}{dk}\right|_0, 0\right) e^{i[k_0(d\omega/dk)|_0 - \omega_0]t} \tag{7.85}$$

This shows that, apart from an overall phase factor, the pulse travels along undistorted in shape with a velocity, called the *group velocity*:

$$v_g = \left.\frac{d\omega}{dk}\right|_0 \tag{7.86}$$

If an energy density is associated with the magnitude of the wave (or its absolute square), it is clear that in this approximation the transport of energy occurs with the group velocity, since that is the rate at which the pulse travels along.

For light waves the relation between ω and k is given by

$$\omega(k) = \frac{ck}{n(k)} \tag{7.87}$$

where c is the velocity of light in vacuum, and $n(k)$ is the index of refraction expressed as a function of k. The phase velocity is

$$v_p = \frac{\omega(k)}{k} = \frac{c}{n(k)} \tag{7.88}$$

and is greater or smaller than c depending on whether $n(k)$ is smaller or larger than unity. For most optical wavelengths $n(k)$ is greater than unity in almost all substances. The group velocity (7.86) is

$$v_g = \frac{c}{n(\omega) + \omega(dn/d\omega)} \tag{7.89}$$

In this equation it is more convenient to think of n as a function of ω than of k. For normal dispersion $(dn/d\omega) > 0$, and also $n > 1$; then the velocity of energy flow is less than the phase velocity and also less than c. In regions of anomalous dispersion, however, $dn/d\omega$ can become large and negative as can be inferred from Fig. 7.8. Then the group velocity differs greatly from the phase velocity, often becoming larger than c or even negative. The behavior of group and phase velocities as a function of frequency in the neighborhood of a region of anomalous dispersion is shown in Fig. 7.14. There is no cause for alarm that our ideas of special relativity are violated; group velocity is *generally* not a useful concept in regions of anomalous dispersion. In addition to the existence of significant absorption (see Fig. 7.8), a large $dn/d\omega$ is equivalent to a rapid variation of ω with k. Consequently the approximations made in (7.83) and following equations are no longer valid. Usually a pulse with its dominant frequency components in the neighborhood of a strong absorption line is absorbed and distorted as it travels. As shown by Garret and McCumber,* however, there are circumstances

*C. G. B. Garrett and D. E. McCumber, *Phys. Rev. A* **1**, 305 (1970).

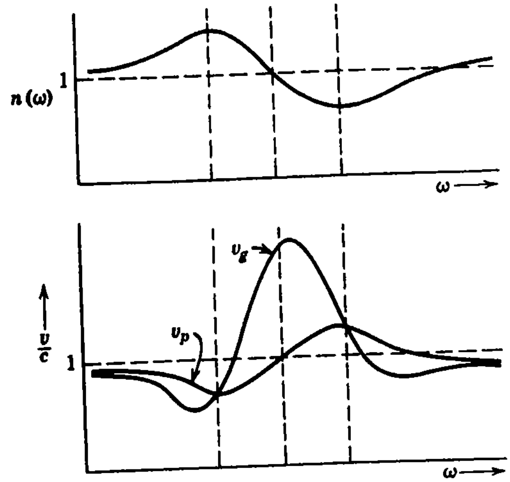

Figure 7.14 Index of refraction $n(\omega)$ as a function of frequency ω at a region of anomalous dispersion; phase velocity v_p and group velocity v_g as functions of ω.

in which "group velocity" can still have meaning, even with anomalous dispersion. Other authors* subsequently verified experimentally what Garrett and McCumber showed theoretically: namely, if absorbers are not too thick, a Gaussian pulse with a central frequency near an absorption line and with support narrow compared to the width of the line (pulse wide in time compared to $1/\gamma$) propagates with appreciable absorption, but more or less retains its shape, the peak of which moves at the group velocity (7.89), even when that quantity is negative. Physically, what occurs is pulse reshaping—the leading edge of the pulse is less attenuated than the trailing edge. Conditions can be such that the *peak* of the greatly attenuated pulse emerges from the absorber before the *peak* of the incident pulse has entered it! (That is the meaning of negative group velocity.) Since a Gaussian pulse does not have a sharply defined front edge, there is no question of violation of causality.

Some experiments are described as showing that photons travel faster than the speed of light through optical "band-gap" devices that reflect almost all of the incident flux over a restricted range of frequencies. While it is true that the centroid of the very small transmitted Gaussian pulse appears slightly in advance of the vacuum transit time, no signal or information travels faster than c. The main results are explicable in conventional classical terms. Some aspects are examined in Problems 7.9–7.11. A review of these and other experiments has been given by Chiao and Steinberg.†

7.9 *Illustration of the Spreading of a Pulse as It Propagates in a Dispersive Medium*

To illustrate the ideas of the preceding section and to show the validity of the concept of group velocity, we now consider a specific model for the dependence

*S. Chu and S. Wong, *Phys. Rev. Letters* **48**, 738 (1982); A. Katz, R. R. Alfano, S. Chu, and S. Wong, *Phys. Rev. Letters* **49**, 1292 (1982).

†R. Y. Chiao and A. M. Steinberg, in *Progress in Optics*, Vol. 37, ed. E. Wolf, Elsevier, Amsterdam (1997), p. 347–406.

of frequency on wave number and calculate without approximations the propagation of a pulse in this model medium. Before specifying the particular model it is necessary to state the initial-value problem in more detail than was done in (7.80) and (7.81). As noted there, the proper specification of an initial-value problem for the wave equation demands the initial values of both function $u(x, 0)$ and time derivative $\partial u(x, 0)/\partial t$. If we agree to take the real part of (7.80) to obtain $u(x, t)$,

$$u(x, t) = \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{ikx-i\omega(k)t}\,dk + \text{c.c.} \tag{7.90}$$

then it is easy to show that $A(k)$ is given in terms of the initial values by:

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\left[u(x, 0) + \frac{i}{\omega(k)}\frac{\partial u}{\partial t}(x, 0)\right] dx \tag{7.91}$$

We take a Gaussian modulated oscillation

$$u(x, 0) = e^{-x^2/2L^2}\cos k_0 x \tag{7.92}$$

as the initial shape of the pulse. For simplicity, we will assume that

$$\frac{\partial u}{\partial t}(x, 0) = 0 \tag{7.93}$$

This means that at times immediately before $t = 0$ the wave consisted of two pulses, both moving toward the origin, such that at $t = 0$ they coalesced into the shape given by (7.92). Clearly at later times we expect each pulse to reemerge on the other side of the origin. Consequently the initial distribution (7.92) may be expected to split into two identical packets, one moving to the left and one to the right. The Fourier amplitude $A(k)$ for the pulse described by (7.92) and (7.93) is

$$\begin{aligned} A(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}e^{-x^2/2L^2}\cos k_0 x\,dx \\ &= \frac{L}{2}\left[e^{-(L^2/2)(k-k_0)^2} + e^{-(L^2/2)(k+k_0)^2}\right] \end{aligned} \tag{7.94}$$

The symmetry $A(-k) = A(k)$ is a reflection of the presence of two pulses traveling away from the origin, as is seen below.

To calculate the waveform at later times, we must specify $\omega = \omega(k)$. As a model allowing exact calculation and showing the essential dispersive effects, we assume

$$\omega(k) = \nu\left(1 + \frac{a^2k^2}{2}\right) \tag{7.95}$$

where ν is a constant frequency, and a is a constant length that is a typical wavelength where dispersive effects become important. Equation (7.95) is an approximation to the dispersion equation of the tenuous plasma, (7.59) or (7.61). Since the pulse (7.92) is a modulated wave of wave number $k = k_0$, the approximate arguments of the preceding section imply that the two pulses will travel with the group velocity

$$v_g = \frac{d\omega}{dk}(k_0) = \nu a^2 k_0 \tag{7.96}$$

and will be essentially unaltered in shape provided the pulse is not too narrow in space.

The exact behavior of the wave as a function of time is given by (7.90), with (7.94) for $A(k)$:

$$u(x, t) = \frac{L}{2\sqrt{2\pi}} \text{Re} \int_{-\infty}^{\infty} [e^{-(L^2/2)(k-k_0)^2} + e^{-(L^2/2)(k+k_0)^2}]\, e^{ikx - i\nu t[1+(a^2k^2/2)]}\, dk \tag{7.97}$$

The integrals can be performed by appropriately completing the squares in the exponents. The result is

$$u(x, t) = \frac{1}{2}\text{Re}\left\{ \frac{\exp\left[-\dfrac{(x - \nu a^2 k_0 t)^2}{2L^2\left(1 + \dfrac{ia^2\nu t}{L^2}\right)}\right]}{\left(1 + \dfrac{ia^2\nu t}{L^2}\right)^{1/2}} \exp\left[ik_0x - i\nu\left(1 + \frac{a^2k_0^2}{2}\right)t\right] + (k_0 \to -k_0) \right\} \tag{7.98}$$

Equation (7.98) represents two pulses traveling in opposite directions. The peak amplitude of each pulse travels with the group velocity (7.96), while the modulation envelop remains Gaussian in shape. The width of the Gaussian is not constant, however, but increases with time. The width of the envelope is

$$L(t) = \left[L^2 + \left(\frac{a^2\nu t}{L}\right)^2\right]^{1/2} \tag{7.99}$$

Thus the dispersive effects on the pulse are greater (for a given elapsed time), the sharper the envelope. The criterion for a small change in shape is that $L \gg a$. Of course, at long times the width of the Gaussian increases linearly with time

$$L(t) \to \frac{a^2\nu t}{L} \tag{7.100}$$

but the time of attainment of this asymptotic form depends on the ratio (L/a). A measure of how rapidly the pulse spreads is provided by a comparison of $L(t)$ given by (7.99), with $v_g t = \nu a^2 k_0 t$. Figure 7.15 shows two examples of curves of the position of peak amplitude $(v_g t)$ and the positions $v_g t \pm L(t)$, which indicate the spread of the pulse, as functions of time. On the left the pulse is not too narrow compared to the wavelength k_0^{-1} and so does not spread too rapidly. The

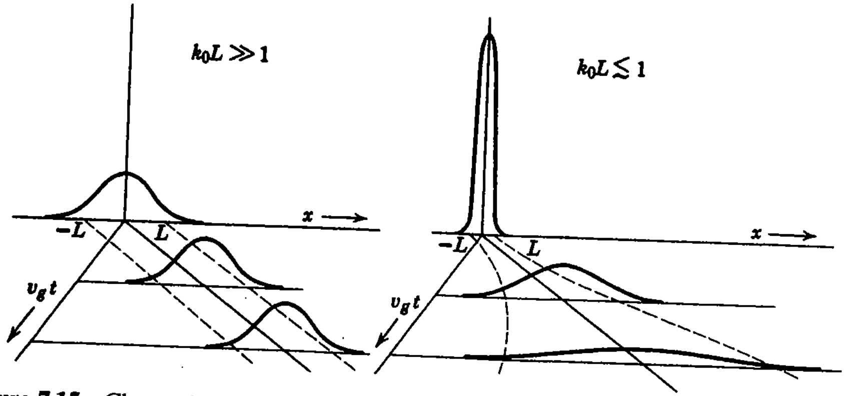

Figure 7.15 Change in shape of a wave packet as it travels along. The broad packet, containing many wavelengths ($k_0L \gg 1$), is distorted comparatively little, while the narrow packet ($k_0L \lesssim 1$) broadens rapidly.

pulse on the right, however, is so narrow initially that it is very rapidly spread out and scarcely represents a pulse after a short time.

Although the results above have been derived for a special choice (7.92) of initial pulse shape and dispersion relation (7.95), their implications are of a more general nature. We saw in Section 7.8 that the average velocity of a pulse is the group velocity $v_g = d\omega/dk = \omega'$. The spreading of the pulse can be accounted for by noting that a pulse with an initial spatial width Δx_0 must have inherent in it a spread of wave numbers $\Delta k \sim (1/\Delta x_0)$. This means that the group velocity, when evaluated for various k values within the pulse, has a spread in it of the order

$$\Delta v_g \sim \omega'' \, \Delta k \sim \frac{\omega''}{\Delta x_0} \tag{7.101}$$

At a time t this implies a spread in position of the order of $\Delta v_g t$. If we combine the uncertainties in position by taking the square root of the sum of squares, we obtain the width $\Delta x(t)$ at time t:

$$\Delta x(t) \simeq \sqrt{(\Delta x_0)^2 + \left(\frac{\omega'' t}{\Delta x_0}\right)^2} \tag{7.102}$$

We note that (7.102) agrees exactly with (7.99) if we put $\Delta x_0 = L$. The expression (7.102) for $\Delta x(t)$ shows the general result that, if $\omega'' \neq 0$, a narrow pulse spreads rapidly because of its broad spectrum of wave numbers, and vice versa. All these ideas carry over immediately into wave mechanics. They form the basis of the Heisenberg uncertainty principle. In wave mechanics, the frequency is identified with energy divided by Planck's constant, while wave number is momentum divided by Planck's constant.

The problem of wave packets in a dissipative, as well as dispersive, medium is rather complicated. Certain aspects can be discussed analytically, but the analytical expressions are not readily interpreted physically. Except in special circumstances, wave packets are attenuated and distorted appreciably as they propagate. The reader may refer to *Stratton* (pp. 301–309) for a discussion of the problem, including numerical examples.

7.10 Causality in the Connection Between D and E; Kramers–Kronig Relations

A. Nonlocality in Time

Another consequence of the frequency dependence of $\epsilon(\omega)$ is a temporally nonlocal connection between the displacement $\mathbf{D}(\mathbf{x}, t)$ and the electric field $\mathbf{E}(\mathbf{x}, t)$. If the monochromatic components of frequency ω are related by

$$\mathbf{D}(\mathbf{x}, \omega) = \epsilon(\omega)\mathbf{E}(\mathbf{x}, \omega) \tag{7.103}$$

the dependence on time can be constructed by Fourier superposition. Treating the spatial coordinate as a parameter, the Fourier integrals in time and frequency can be written

$$\mathbf{D}(\mathbf{x}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{D}(\mathbf{x}, \omega) e^{-i\omega t}\, d\omega$$

and

$$\mathbf{D}(\mathbf{x}, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{D}(\mathbf{x}, t') e^{i\omega t'}\, dt' \tag{7.104}$$

with corresponding equations for $\mathbf{E}$. The substitution of (7.103) for $\mathbf{D}(\mathbf{x}, \omega)$ gives

$$\mathbf{D}(\mathbf{x}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \epsilon(\omega)\mathbf{E}(\mathbf{x}, \omega) e^{-i\omega t}\, d\omega$$

We now insert the Fourier representation of $\mathbf{E}(\mathbf{x}, \omega)$ into the integral and obtain

$$\mathbf{D}(\mathbf{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, \epsilon(\omega) e^{-i\omega t} \int_{-\infty}^{\infty} dt'\, e^{i\omega t'} \mathbf{E}(\mathbf{x}, t')$$

With the assumption that the orders of integration can be interchanged, the last expression can be written as

$$\mathbf{D}(\mathbf{x}, t) = \epsilon_0 \left\{ \mathbf{E}(\mathbf{x}, t) + \int_{-\infty}^{\infty} G(\tau)\mathbf{E}(\mathbf{x}, t - \tau)\, d\tau \right\} \tag{7.105}$$

where $G(\tau)$ is the Fourier transform of $\chi_e = \epsilon(\omega)/\epsilon_0 - 1$:

$$G(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\epsilon(\omega)/\epsilon_0 - 1] e^{-i\omega\tau}\, d\omega \tag{7.106}$$

Equations (7.105) and (7.106) give a nonlocal connection between $\mathbf{D}$ and $\mathbf{E}$, in which $\mathbf{D}$ at time t depends on the electric field at times other than t.* If $\epsilon(\omega)$ is

*Equations (7.103) and (7.105) are recognizable as an example of the *faltung* theorem of Fourier integrals: if $A(t)$, $B(t)$, $C(t)$ and $a(\omega)$, $b(\omega)$, $c(\omega)$ are two sets of functions related in pairs by the Fourier inversion formulas (7.104), and

$$c(\omega) = a(\omega)b(\omega)$$

then, under suitable restrictions concerning integrability,

$$C(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(t')B(t - t')\, dt'$$

independent of ω for all ω, (7.106) yields $G(\tau) \propto \delta(\tau)$ and the instantaneous connection is obtained, but if $\epsilon(\omega)$ varies with ω, $G(\tau)$ is nonvanishing for some values of τ different from zero.

B. Simple Model for $G(\tau)$, Limitations

To illustrate the character of the connection implied by (7.105) and (7.106) we consider a one-resonance version of the index of refraction (7.51):

$$\epsilon(\omega)/\epsilon_0 - 1 = \omega_p^2(\omega_0^2 - \omega^2 - i\gamma\omega)^{-1} \tag{7.107}$$

The susceptibility kernel $G(\tau)$ for this model of $\epsilon(\omega)$ is

$$G(\tau) = \frac{\omega_p^2}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\omega\tau}}{\omega_0^2 - \omega^2 - i\gamma\omega}\, d\omega \tag{7.108}$$

The integral can be evaluated by contour integration. The integrand has poles in the lower half-ω-plane at

$$\omega_{1,2} = -\frac{i\gamma}{2} \pm \nu_0, \qquad \text{where } \nu_0^2 = \omega_0^2 - \frac{\gamma^2}{4} \tag{7.109}$$

For $\tau < 0$ the contour can be closed in the upper half-plane without affecting the value of the integral. Since the integrand is regular inside the closed contour, the integral vanishes. For $\tau > 0$, the contour is closed in the lower half-plane and the integral is given by $-2\pi i$ times the residues at the two poles. The kernel (7.108) is therefore

$$G(\tau) = \omega_p^2 e^{-\gamma\tau/2}\, \frac{\sin \nu_0\tau}{\nu_0}\, \theta(\tau) \tag{7.110}$$

where $\theta(\tau)$ is the step function [$\theta(\tau) = 0$ for $\tau < 0$; $\theta(\tau) = 1$ for $\tau > 0$]. For the dielectric constant (7.51) the kernel $G(\tau)$ is just a linear superposition of terms like (7.110). The kernel $G(\tau)$ is oscillatory with the characteristic frequency of the medium and damped in time with the damping constant of the electronic oscillators. The nonlocality in time of the connection between **D** and **E** is thus confined to times of the order of γ^{-1}. Since γ is the width in frequency of spectral lines and these are typically 10^7–10^9 s^{-1}, the departure from simultaneity is of the order of 10^{-7}–10^{-9} s. For frequencies above the microwave region many cycles of the electric field oscillations contribute an average weighed by $G(\tau)$ to the displacement **D** at a given instant of time.

Equation (7.105) is nonlocal in time, but not in space. This approximation is valid provided the spatial variation of the applied fields has a scale that is large compared with the dimensions involved in the creation of the atomic or molecular polarization. For bound charges the latter scale is of the order of atomic dimensions or less, and so the concept of a dielectric constant that is a function only of ω can be expected to hold for frequencies well beyond the visible range. For conductors, however, the presence of free charges with macroscopic mean free paths makes the assumption of a simple $\epsilon(\omega)$ or $\sigma(\omega)$ break down at much lower frequencies. For a good conductor like copper we have seen that the damping constant (corresponding to a collision frequency) is of the order of $\gamma_0 \sim 3 \times 10^{13}$ s^{-1} at room temperature. At liquid helium temperatures, the damping constant may be 10^{-3} times the room temperature value. Taking the Bohr velocity in hydrogen ($c/137$) as typical of electron velocities in metals, we find mean free paths of the order $L \sim c/(137\gamma_0) \sim 10^{-4}$ m at liquid helium temperatures. On the other hand, the conventional skin depth δ (7.77) can be much smaller, of the order of 10^{-7} or 10^{-8} m at microwave frequencies. In such circumstances, Ohm's law must be replaced by a nonlocal expression. The conductivity becomes a tensorial quantity depending on wave number $\mathbf{k}$ and frequency ω. The associated departures from the standard behavior are known collectively as the *anomalous skin effect.* They can be utilized to map out the Fermi surfaces in metals.* Similar nonlocal effects occur in superconductors where the electromagnetic properties involve a coherence length of the order of 10^{-6} m.† With this brief mention of the limitations of (7.105) and the areas where generalizations have been fruitful we return to the discussion of the physical content of (7.105).

C. Causality and Analyticity Domain of $\epsilon(\omega)$

The most obvious and fundamental feature of the kernel (7.110) is that it vanishes for $\tau < 0$. This means that at time t only values of the electric field *prior* to that time enter in determining the displacement, in accord with our fundamental ideas of causality in physical phenomena. Equation (7.105) can thus be written

$$\mathbf{D}(\mathbf{x}, t) = \epsilon_0\left\{\mathbf{E}(\mathbf{x}, t) + \int_0^\infty G(\tau)\mathbf{E}(\mathbf{x}, t - \tau)\, d\tau\right\} \tag{7.111}$$

This is, in fact, the most general spatially local, linear, and causal relation that can be written between $\mathbf{D}$ and $\mathbf{E}$ in a uniform isotropic medium. Its validity transcends any specific model of $\epsilon(\omega)$. From (7.106) the dielectric constant can be expressed in terms of $G(\tau)$ as

$$\epsilon(\omega)/\epsilon_0 = 1 + \int_0^\infty G(\tau)e^{i\omega\tau}\, d\tau \tag{7.112}$$

This relation has several interesting consequences. From the reality of $\mathbf{D}$, $\mathbf{E}$, and therefore $G(\tau)$ in (7.111) we can deduce from (7.112) that for complex ω,

$$\epsilon(-\omega)/\epsilon_0 = \epsilon^*(\omega^*)/\epsilon_0 \tag{7.113}$$

Furthermore, if (7.112) is viewed as a representation of $\epsilon(\omega)/\epsilon_0$ in the complex ω plane, it shows that *$\epsilon(\omega)/\epsilon_0$ is an analytic function of ω in the upper half-plane,* provided $G(\tau)$ is finite for all τ. On the real axis it is necessary to invoke the "physically reasonable" requirement that $G(\tau) \to 0$ as $\tau \to \infty$ to assure that $\epsilon(\omega)/\epsilon_0$ is also analytic there. This is true for dielectrics, but not for conductors, where $G(\tau) \to \sigma/\epsilon_0$ as $\tau \to \infty$ and $\epsilon(\omega)/\epsilon_0$ has a simple pole at $\omega = 0$ ($\epsilon \to i\sigma/\omega$ as $\omega \to 0$). Apart, then, from a possible pole at $\omega = 0$, the dielectric constant $\epsilon(\omega)/\epsilon_0$ is analytic in ω for Im $\omega \geq 0$ as a direct result of the causal relation (7.111)

*A. B. Pippard, in *Reports on Progress in Physics* **23**, 176 (1960), and the article entitled "The Dynamics of Conduction Electrons," by the same author in *Low-Temperature Physics*, Les Houches Summer School (1961), eds. C. de Witt, B. Dreyfus, and P. G. de Gennes, Gordon and Breach, New York (1962). The latter article has been issued separately by the same publisher.

†See, for example, the article "Superconductivity" by M. Tinkham in *Low Temperature Physics, op. cit.*

between **D** and **E**. These properties can be verified, of course, for the models discussed in Sections 7.5.A and 7.5.C.

The behavior of $\epsilon(\omega)/\epsilon_0 - 1$ for large ω can be related to the behavior of $G(\tau)$ at small times. Integration by parts in (7.112) leads to the asymptotic series,

$$\epsilon(\omega)/\epsilon_0 - 1 \simeq \frac{iG(0)}{\omega} - \frac{G'(0)}{\omega^2} + \cdots$$

where the argument of G and its derivatives is $\tau = 0^+$. It is unphysical to have $G(0^-) = 0$, but $G(0^+) \neq 0$. Thus the first term in the series is absent, and $\epsilon(\omega)/\epsilon_0 - 1$ falls off at high frequencies as ω^{-2}, just as was found in (7.59) for the oscillator model. The asymptotic series shows, in fact, that the real and imaginary parts of $\epsilon(\omega)/\epsilon_0 - 1$ behave for large real ω as

$$\text{Re}[\epsilon(\omega)/\epsilon_0 - 1] = O\left(\frac{1}{\omega^2}\right), \qquad \text{Im } \epsilon(\omega)/\epsilon_0 = O\left(\frac{1}{\omega^3}\right) \tag{7.114}$$

These asymptotic forms depend only upon the existence of the derivatives of $G(\tau)$ around $\tau = 0^+$.

D. *Kramers–Kronig Relations*

The analyticity of $\epsilon(\omega)/\epsilon_0$ in the upper half-ω-plane permits the use of Cauchy's theorem to relate the real and imaginary part of $\epsilon(\omega)/\epsilon_0$ on the real axis. For any point z inside a closed contour C in the upper half-ω-plane, Cauchy's theorem gives

$$\epsilon(z)/\epsilon_0 = 1 + \frac{1}{2\pi i}\oint_C \frac{[\epsilon(\omega')/\epsilon_0 - 1]}{\omega' - z}\, d\omega'$$

The contour C is now chosen to consist of the real ω axis and a great semicircle at infinity in the upper half-plane. From the asymptotic expansion just discussed or the specific results of Section 7.5.D, we see that $\epsilon/\epsilon_0 - 1$ vanishes sufficiently rapidly at infinity so that there is no contribution to the integral from the great semicircle. Thus the Cauchy integral can be written

$$\epsilon(z)/\epsilon_0 = 1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{[\epsilon(\omega')/\epsilon_0 - 1]}{\omega' - z}\, d\omega' \tag{7.115}$$

where z is now any point in the upper half-plane and the integral is taken along the real axis. Taking the limit as the complex frequency approaches the real axis from above, we write $z = \omega + i\delta$ in (7.115):

$$\epsilon(\omega)/\epsilon_0 = 1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{[\epsilon(\omega')/\epsilon_0 - 1]}{\omega' - \omega - i\delta}\, d\omega' \tag{7.116}$$

For real ω the presence of the $i\delta$ in the denominator is a mnemonic for the distortion of the contour along the real axis by giving it an infinitesimal semicircular detour *below* the point $\omega' = \omega$. The denominator can be written formally as

$$\frac{1}{\omega' - \omega - i\delta} = P\left(\frac{1}{\omega' - \omega}\right) + \pi i\delta(\omega' - \omega) \tag{7.117}$$

where P means principal part. The delta function serves to pick up the contribution from the small semicircle going in a positive sense halfway around the pole at $\omega' = \omega$. Use of (7.117) and a simple rearrangement turns (7.116) into

$$\epsilon(\omega)/\epsilon_0 = 1 + \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{[\epsilon(\omega')/\epsilon_0 - 1]}{\omega' - \omega} d\omega' \tag{7.118}$$

The real and imaginary parts of this equation are

$$\begin{aligned} \text{Re}\ \epsilon(\omega)/\epsilon_0 &= 1 + \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\text{Im}\ \epsilon(\omega')/\epsilon_0}{\omega' - \omega} d\omega' \\ \text{Im}\ \epsilon(\omega)/\epsilon_0 &= -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{[\text{Re}\ \epsilon(\omega')/\epsilon_0 - 1]}{\omega' - \omega} d\omega' \end{aligned} \tag{7.119}$$

These relations, or the ones recorded immediately below, are called *Kramers–Kronig relations* or *dispersion relations*. They were first derived by H. A. Kramers (1927) and R. de L. Kronig (1926) independently. The symmetry property (7.113) shows that Re $\epsilon(\omega)$ is even in ω, while Im $\epsilon(\omega)$ is odd. The integrals in (7.119) can thus be transformed to span only positive frequencies:

$$\begin{aligned} \text{Re}\ \epsilon(\omega)/\epsilon_0 &= 1 + \frac{2}{\pi} P \int_{0}^{\infty} \frac{\omega'\ \text{Im}\ \epsilon(\omega')/\epsilon_0}{\omega'^2 - \omega^2} d\omega' \\ \text{Im}\ \epsilon(\omega)/\epsilon_0 &= -\frac{2\omega}{\pi} P \int_{0}^{\infty} \frac{[\text{Re}\ \epsilon(\omega')/\epsilon_0 - 1]}{\omega'^2 - \omega^2} d\omega' \end{aligned} \tag{7.120}$$

In writing (7.119) and (7.120) we have tacitly assumed that $\epsilon(\omega)/\epsilon_0$ was regular at $\omega = 0$. For conductors the simple pole at $\omega = 0$ can be exhibited separately with little further complication.

The Kramers–Kronig relations are of very general validity, following from little more than the assumption of the causal connection (7.111) between the polarization and the electric field. Empirical knowledge of Im $\epsilon(\omega)$ from absorption studies allows the calculation of Re $\epsilon(\omega)$ from the first equation in (7.120). The connection between absorption and anomalous dispersion, shown in Fig. 7.8, is contained in the relations. The presence of a very narrow absorption line or band at $\omega = \omega_0$ can be approximated by taking

$$\text{Im}\ \epsilon(\omega') \simeq \frac{\pi K}{2\omega_0} \delta(\omega' - \omega_0) + \cdots$$

where K is a constant and the dots indicate the other (smoothly varying) contributions to Im ϵ. The first equation in (7.120) then yields

$$\text{Re}\ \epsilon(\omega) \simeq \bar{\epsilon} + \frac{K}{\omega_0^2 - \omega^2} \tag{7.121}$$

for the behavior of Re $\epsilon(\omega)$ near, but not exactly at, $\omega = \omega_0$. The term $\bar{\epsilon}$ represents the slowly varying part of Re ϵ resulting from the more remote contributions to Im ϵ. The approximation (7.121) exhibits the rapid variation of Re $\epsilon(\omega)$ in the neighborhood of an absorption line, shown in Fig. 7.8 for lines of finite width. A more realistic description for Im ϵ would lead to an expression for Re ϵ in complete accord with the behavior shown in Fig. 7.8. The demonstration of this is left to the problems at the end of the chapter.

Relations of the general type (7.119) or (7.120) connecting the dispersive and absorptive aspects of a process are extremely useful in all areas of physics. Their widespread application stems from the very small number of physically well-founded assumptions necessary for their derivation. References to their application in particle physics, as well as solid-state physics, are given at the end of the chapter. We end with mention of two *sum rules* obtainable from (7.120). It was shown in Section 7.5.D, within the context of a specific model, that the dielectric constant is given at high frequencies by (7.59). The form of (7.59) is, in fact, quite general, as shown above (Section 7.10.C). The plasma frequency can therefore be *defined* by means of (7.59) as

$$\omega_p^2 = \lim_{\omega\to\infty}\{\omega^2[1 - \epsilon(\omega)/\epsilon_0]\}$$

Provided the falloff of Im $\epsilon(\omega)$ at high frequencies is given by (7.114), the first Kramers–Kronig relation yields a *sum rule for* ω_p^2:

$$\omega_p^2 = \frac{2}{\pi}\int_0^\infty \omega \text{ Im } \epsilon(\omega)/\epsilon_0 \, d\omega \tag{7.122}$$

This relation is sometimes known as the sum rule for oscillator strengths. It can be shown to be equivalent to (7.52) for the dielectric constant (7.51), but is obviously more general.

The second sum rule concerns the integral over the real part of $\epsilon(\omega)$ and follows from the second relation (7.120). With the assumption that $[\text{Re } \epsilon(\omega')/\epsilon_0 - 1] = -\omega_p^2/\omega'^2 + O(1/\omega'^4)$ for all $\omega' > N$, it is straightforward to show that for $\omega > N$

$$\text{Im } \epsilon(\omega)/\epsilon_0 = \frac{2}{\pi\omega}\left\{-\frac{\omega_p^2}{N} + \int_0^N [\text{Re } \epsilon(\omega')/\epsilon_0 - 1]\, d\omega'\right\} + O\left(\frac{1}{\omega^3}\right)$$

It was shown in Section 7.10.C that, excluding conductors and barring the unphysical happening that $G(0^+) \neq 0$, Im $\epsilon(\omega)$ behaves at large frequencies as ω^{-3}. It therefore follows that the expression in curly brackets must vanish. We are thus led to a *second sum rule*,

$$\frac{1}{N}\int_0^N \text{Re } \epsilon(\omega)/\epsilon_0 \, d\omega = 1 + \frac{\omega_p^2}{N^2} \tag{7.123}$$

which, for $N \to \infty$, states that the average value of Re $\epsilon(\omega)/\epsilon_0$ over all frequencies is equal to unity. For conductors, the plasma frequency sum rule (7.122) still holds, but the second sum rule (sometimes called a *superconvergence relation*) has an added term $-\pi\sigma/2\epsilon_0 N$, on the right hand side (see Problem 7.23). These optical sum rules and several others are discussed by Altarelli et al.*

7.11 *Arrival of a Signal After Propagation Through a Dispersive Medium*

Some of the effects of dispersion have been considered in the preceding sections. There remains one important aspect, the actual arrival at a remote point of a

*M. Altarelli, D. L. Dexter, H. M. Nussenzveig, and D. Y. Smith, *Phys. Rev.* **B6**, 4502 (1972).

wave train that initially has a well-defined beginning. How does the signal build up? If the phase velocity or group velocity is greater than the velocity of light in vacuum for important frequency components, does the signal propagate faster than allowed by causality and relativity? Can the arrival time of the disturbance be given an unambiguous definition? These questions were examined authoritatively by Sommerfeld and Brillouin in papers published in *Annalen der Physik* in 1914.* The original papers, plus subsequent work by Brillouin, are contained in English translation in the book, *Wave Propagation and Group Velocity*, by Brillouin. A briefer account is given in Sommerfeld's *Optics*, Chapter III. A complete discussion is lengthy and technically complicated.† We treat only the qualitative features. The reader can obtain more detail in the cited literature or the second edition of this book, from which the present account is abbreviated.

For definiteness we consider a plane wave train normally incident from vacuum on a semi-infinite uniform medium of index of refraction $n(\omega)$ filling the region $x > 0$. From the Fresnel equations (7.42) and Problem 7.20, the amplitude of the electric field of the wave for $x > 0$ is given by

$$u(x, t) = \int_{-\infty}^{\infty} \left[\frac{2}{1 + n(\omega)}\right] A(\omega) e^{ik(\omega)x - i\omega t} \, d\omega \tag{7.124}$$

where

$$A(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u_i(0, t) e^{i\omega t} \, dt \tag{7.125}$$

is the Fourier transform of the real incident electric field $u_i(x, t)$ evaluated just outside the medium, at $x = 0^-$. The wave number $k(\omega)$ is

$$k(\omega) = \frac{\omega}{c} n(\omega) \tag{7.126}$$

and is generally complex, with positive imaginary part corresponding to absorption of energy during propagation. Many media are sufficiently transparent that the wave number can be treated as real for most purposes, but there is always some damping present. [Parenthetically we observe that in (7.124) frequency, not wave number, is used as the independent variable. The change from the practice of Sections 7.8 and 7.9 is dictated by the present emphasis on the *time* development of the wave at a fixed point in space.]

We suppose that the incident wave has a well-defined front edge that reaches $x = 0$ not before $t = 0$. Thus $u(0, t) = 0$ for $t < 0$. With additional physically reasonable mathematical requirements, this condition on $u(0, t)$ assures that $A(\omega)$ is analytic in the upper half-ω-plane [just as condition (7.112) assured the analyticity of $\epsilon(\omega)$ there]. Generally, $A(\omega)$ will have singularities in the lower half-ω-plane determined by the exact form of $u(x, t)$. We assume that $A(\omega)$ is bounded for $|\omega| \to \infty$.

The index of refraction $n(\omega)$ is crucial in determining the detailed nature of the propagation of the wave in the medium. Some general features follow, how-

*A Sommerfeld, *Ann. Phys* (*Leipzig*) **44**, 177 (1914). L. Brillouin, *Ann. Phys.* (*Leipzig*) **44**, 203 (1914).

†An exhaustive treatment is given in K. E. Oughstun and G. C. Sherman, *Electromagnetic Pulse Propagation in Causal Dielectrics*, Springer-Verlag, Berlin (1994).

ever, from the global properties of $n(\omega)$. Just as $\epsilon(\omega)$ is analytic in the upper half-ω-plane, so is $n(\omega)$. Furthermore, (7.59) shows that for $|\omega| \to \infty$, $n(\omega) \to 1 - \omega_p^2/2\omega^2$. A simple one-resonance model of $n(\omega)$ based on (7.51), with resonant frequency ω_0 and damping constant γ, leads to the singularity structure shown in Fig. 7.16. The poles of $\epsilon(\omega)$ become branch cuts in $n(\omega)$. A multiresonance expression for ϵ leads to a much more complex cut structure, but the upper plane analyticity and the asymptotic behavior for large $|\omega|$ remain.

The proof that no signal can propagate faster than the speed of light in vacuum, whatever the detailed properties of the medium, is now straightforward. We consider evaluating the amplitude (7.124) by contour integration in the complex ω plane. Since $n(\omega) \to 1$ for $|\omega| \to \infty$, the argument of the exponential in (7.124) becomes

$$i\phi(\omega) = i[k(\omega)x - \omega t] \to \frac{i\omega(x - ct)}{c}$$

for large $|\omega|$. Evidently, we obtain a vanishing contribution to the integral by closing the contour with a great semicircle at infinity in the upper half-plane for $x > ct$ and in the lower half-plane for $x < ct$. With $n(\omega)$ and $A(\omega)$ both analytic in the upper half-ω-plane, the whole integrand is analytic there. Cauchy's theorem tells us that if the contour is closed in the upper half-plane ($x > ct$), the integral vanishes. We have therefore established that

$$u(x, t) = 0 \qquad \text{for } (x - ct) > 0 \tag{7.127}$$

provided only that $A(\omega)$ and $n(\omega)$ are analytic for $\text{Im}\,\omega > 0$ and $n(\omega) \to 1$ for $|\omega| \to \infty$. Since the specific form of $n(\omega)$ does not enter, we have a general proof that no signal propagates with a velocity greater than c, whatever the medium.

For $ct > x$, the contour is to be closed in the lower half-plane, enveloping the singularities. The integral is dominated by different singularities at different

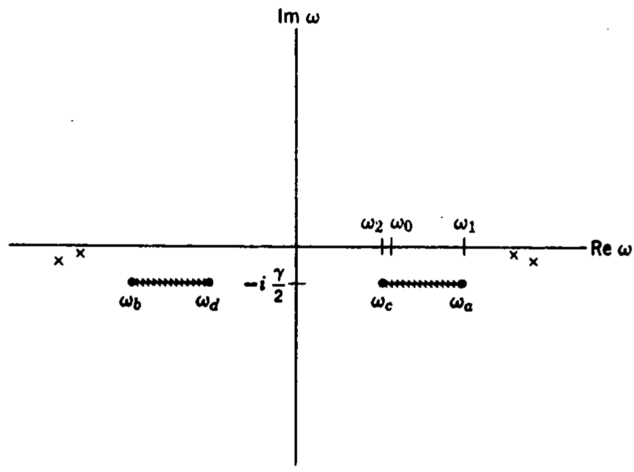

Figure 7.16 Branch cuts defining the singularities of a simple one-resonance model for the index of refraction $n(\omega)$. For transparent media the branch cuts lie much closer to (but still below) the real axis than shown here. More realistic models for $n(\omega)$ have more complicated cut structures, all in the lower half-ω-plane. The crosses mark the possible locations of singularities of $A(\omega)$.